A SOLUTION TO THE BERNSTEIN PROBLEM IN THE THREE-DIMENSIONAL HEISENBERG GROUP VIA LOOP GROUPS

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ABSTRACT. In this note we present a short alternative proof for the Bernstein problem in the three-dimensional Heisenberg group Nil_3 by using the loop group technique.

INTRODUCTION

The *Bernstein problem* is one of the traditional problems of global differential geometry. The original result due to Bernstein asserts that every entire minimal graph in Euclidean three-space $\mathbb{R}^3(x_1, x_2, x_3)$ is a plane. In other words, Bernstein's result shows that the only global solution on the (x_1, x_2)-plane to the so-called *minimal surface equation*

$$\{1 + (f_{x_1})^2\}f_{x_2x_2} - 2f_{x_1}f_{x_2}f_{x_1x_2} + \{1 + (f_{x_2})^2\}f_{x_1x_1} = 0$$

is a linear function of x_1 and x_2.

The Bernstein problem has been generalized to a problem basically asking for a classification of all entire minimal graphs. On the other hand, when the ambient space is not the Euclidean three-space, the Bernstein problem often needs to be amended. For instance, in Minkowski three-space $\mathbb{L}_3$ equipped with the natural Lorentz metric $dx_1^2 + dx_2^2 - dx_3^2$, there are many entire (timelike) minimal graphs over the timelike plane $\mathbb{L}_2 = \mathbb{R}^2(x_2, x_3)$, see for example [9].

Next, we focus on the three-dimensional *Heisenberg group* Nil_3 which is one of the model spaces of Thurston geometries [10]. The space Nil_3 is realized as Cartesian three-space $\mathbb{R}^3(x_1, x_2, x_3)$ equipped with the Riemannian metric

$$dx_1^2 + dx_2^2 + \left\{dx_3 + \frac{1}{2}(x_2dx_1 - x_1dx_2)\right\}^2$$

and a nilpotent Lie group structure, see for example [5]. The Riemannian metric is invariant under the nilpotent Lie group structure and has a 4-dimensional isometry group. The identity component of the isometry group is a semi-direct product $\mathrm{Nil}_3 \ltimes \mathrm{SO}_2$.

It has been known for a long time that in Nil_3 nontrivial entire minimal graphs exist, see for example [6]. Therefore, in Nil_3 the Bernstein problem has been phrased more specifically as the problem to construct entire minimal graphs over the natural (x_1, x_2)-plane with a prescribed holomorphic quadratic differential.

Under this formulation, Fernández and Mira studied the Bernstein problem in Nil_3 [7]. They proved that for a prescribed holomorphic quadratic differential $Q\,dz^2$ over the complex plane $\mathbb{C}$ with $Q \neq 0$ or the unit disc $\mathbb{D}$, there exists a two-parameter family of entire minimal vertical graphs whose Abresch-Rosenberg differential is $Q\,dz^2$. Their proof relies firstly on the Lawson-type correspondence (often called *sister correspondence*) between minimal surfaces in Nil_3 and surfaces of constant mean curvature (CMC in short) with mean curvature $H = 1/2$ in the product space $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ denotes the hyperbolic two-space. Secondly, they use the correspondence between harmonic maps into $\mathbb{H}^2$ and CMC surfaces with mean curvature $H = 1/2$ in the product space $\mathbb{H}^2 \times \mathbb{R}$. Finally, they use a result of Wan and Au [11, 12] solving the Bernstein problem for spacelike CMC surfaces in $\mathbb{L}_3$ and use that the Gauss map of those surfaces is also harmonic into $\mathbb{H}^2$.

Our proof is much more direct. In our previous work [5], we have established a generalized Weierstrass type representation for minimal surfaces in Nil_3. Every simply connected (nowhere vertical) minimal surface is obtained from an extended frame for a harmonic map into the hyperbolic two-space $\mathbb{H}^2$. In this paper we give a short proof of the solution to the Bernstein problem in Nil_3 by virtue of the generalized Weierstrass type representation established in [5]. The advantage of our approach is that we can give a direct relation between minimal graphs in Nil_3 and spacelike CMC surface with mean curvature $H = 1/2$ graphs in $\mathbb{L}_3$, Theorem 1.7. This relation enables us to give a simple alternative proof of Fernández-Mira's theorem, Theorem 1.8.

Our new proof actually also provides new insights. In fact our proof clarifies the geometric meaning of the two-parameter ambiguity of entire minimal graphs with prescribed Abresch-Rosenberg differential. While it is quite clear that the two-parameter family is related to the boosts in $\mathrm{SU}_{1,1}$, our argument also shows how the corresponding family of surfaces varies in Nil_3.

1. Bernstein problem

We discuss the Bernstein problem in Nil_3, that is, the classification of entire minimal vertical graphs in Nil_3. We only consider vertical graphs. Therefore we will sometimes omit the word "vertical". In Appendix B, we give a short review of facts and results of [5] which are used for the solution to the Bernstein problem via loop groups. From now on, we denote the coordinates of Nil_3 or $\mathbb{L}_3$ by (x_1, x_2, x_3).

1.1. **Completeness.** The basic result used in this paper is Theorem B.2. It explains the direct relation between minimal surfaces in Nil_3 and spacelike CMC surfaces in $\mathbb{L}_3$. This close relationship is also underlined by a simple relation between the corresponding metrics.

Lemma 1.1. *Let f^λ and $f^\lambda_{\mathbb{L}_3}$ be an associated family of minimal surfaces in Nil_3 and an associated family of spacelike CMC surfaces with mean curvature $H = 1/2$ in $\mathbb{L}_3$ correlated and defined as in Theorem B.2, respectively. Denote the metric of f^λ by $e^u dz d\bar{z}$ and the metric of $f^\lambda_{\mathbb{L}_3}$ by $e^{u_{\mathbb{L}_3}} dz d\bar{z}$, respectively. Moreover, let $\phi^\lambda_3 dz$ be the coefficient of e_3 in $(f^\lambda)^{-1} f^\lambda_z dz = \sum_{i=1}^{3} (\phi^\lambda_i dz) e_i$. Then the following relation holds:*

$$e^{u_{\mathbb{L}_3}} + 4|\phi^\lambda_3|^2 = e^u.$$

2

Proof. It is known that the conformal factors e^u and $e^{u_{\mathbb{L}_3}}$ can be computed explicitly in terms of spinors, see [5, Section 3.1], Remark B.3 and [2]:

$$e^u = 4(|\psi_1^\lambda|^2 + |\psi_2^\lambda|^2)^2, \quad e^{u_{\mathbb{L}_3}} = 4(|\psi_1^\lambda|^2 - |\psi_2^\lambda|^2)^2,$$

where ψ_j^λ $(j = 1, 2)$ is a family of spinors for the associated family f^λ. Since $\phi_3^\lambda = 2\psi_1^\lambda\overline{\psi_2^\lambda}$, the claim follows. $\qquad\square$

Remark 1.2. It is known that the metrics $e^{u_{\mathbb{L}_3}} dz d\bar{z}$ of an associated family of spacelike CMC surfaces $f_{\mathbb{L}_3}^\lambda$ are independent of λ, that is, on a simply connected domain, any two members of the associated family $\{f_{\mathbb{L}_3}^\lambda\}_{\lambda\in\mathbb{S}^1}$ are isometric. In fact the metric can be computed by the support $h(dz)^{1/2}(d\bar{z})^{1/2}$, see Appendix B for definition, as

$$(1.1) \qquad\qquad h^2 dz d\bar{z} = e^{u_{\mathbb{L}_3}} dz d\bar{z}.$$

However, the metrics $e^u dz d\bar{z}$ of an associated family of minimal surface f^λ depend on λ, that is, any two members of the associated family $\{f^\lambda\}_{\lambda\in\mathbb{S}^1}$ are, in general, non-isometric.

Using the relation above, we have the following theorem.

Theorem 1.3. *Let f^λ and $f_{\mathbb{L}_3}^\lambda$ be an associated family of minimal surfaces in Nil_3 and an associated family of spacelike CMC surfaces with mean curvature $H = 1/2$ in $\mathbb{L}_3$ correlated and defined as in Theorem B.2, respectively. Assume that one member of the associated family $\{f_{\mathbb{L}_3}^\lambda\}_{\lambda\in\mathbb{S}^1}$ is closed with respect to the Euclidean topology. Then each member of the associated family $\{f^\lambda\}_{\lambda\in\mathbb{S}^1}$ is a complete, entire graph.*

Proof. We denote the spacelike CMC surface in $\mathbb{L}_3$ which is closed with respect to the Euclidean topology by $f_{\mathbb{L}_3}^* = f_{\mathbb{L}_3}^\lambda|_{\lambda^*\in\mathbb{S}^1}$. From the assumption and by [3, p. 415], we conclude that $f_{\mathbb{L}_3}^*$ is complete. Moreover from [11, Proposition 2], $f_{\mathbb{L}_3}^*$ is also an entire graph. Since within the associated family $\{f_{\mathbb{L}_3}^\lambda\}_{\lambda\in\mathbb{S}^1}$ the metric is invariant, see Remark 1.2, each member of the associated family of spacelike CMC surfaces $\{f_{\mathbb{L}_3}^\lambda\}_{\lambda\in\mathbb{S}^1}$ is also complete. Then from Lemma 1.1, we have that each member of the associated family of minimal surfaces $\{f^\lambda\}_{\lambda\in\mathbb{S}^1}$ is complete.

Let us look more closely at the correspondence between $f_{\mathbb{L}_3}^\lambda$ and f^λ. From formulas (B.6) and (B.7) we infer by inspection that f^λ and $f_{\mathbb{L}_3}^\lambda$ share the same x_1-, x_2-components. Here, as pointed out before, we denote the coordinates of Nil_3 or $\mathbb{L}_3$ by (x_1, x_2, x_3). Therefore, since $f_{\mathbb{L}_3}^\lambda$ is an entire graph, thus f^λ also is an entire graph. This completes the proof. $\quad\square$

1.2. **Rigid motions.** It is known that the isometry group of $\mathbb{L}_3$ is the six-dimensional Lie group which is generated by a one-parameter family of rotations around x_3-axis (the timelike axis), a two-parameter family of boosts and three families of translations. In contrast, the isometry group of Nil_3 is only four-dimensional and is generated by a one-parameter family of rotations around x_3-axis and three families of translations in Nil_3.

A comparison of the two Sym formulas in Theorem B.2 indicates that isometries of Minkowski space will not necessarily become isometries of Nil_3. The precise relation will be made clear in the Lemma below.

Lemma 1.4. *Let $f_{\mathbb{L}_3}^\lambda$ and $\tilde{f}_{\mathbb{L}_3}^\lambda$ be two associated family of spacelike CMC surfaces with mean curvature $H = 1/2$ in $\mathbb{L}_3$ defined by the Sym-formula in Theorem B.2 for some extended*

3

frames F^λ and $\tilde{F}^\lambda$, respectively and set $f_{\mathbb{L}_3} = f^\lambda_{\mathbb{L}_3}|_{\lambda=1}$ and $\tilde{f}_{\mathbb{L}_3} = \tilde{f}^\lambda_{\mathbb{L}_3}|_{\lambda=1}$. Moreover, let f^λ and $\tilde{f}^\lambda$ denote the two associated families of minimal surfaces in Nil_3 defined from the same extended frames F^λ and $\tilde{F}^\lambda$, respectively and set $f = f^\lambda|_{\lambda=1}$ and $\tilde{f} = \tilde{f}^\lambda|_{\lambda=1}$. Assume that $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ are isometric by some rigid motion in $\mathbb{L}_3$. Then the following statements hold:

(1) *If $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ are isometric by a rotation around x_3-axis (the timelike axis), then f and $\tilde{f}$ are isometric by the rotation around x_3-axis (the same angle) and some translation.*

(2) *If $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ are isometric by a translation, then f and $\tilde{f}$ are isometric by some translation (not necessarily the same translation).*

(3) *If $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ are isometric by a boost, then f and $\tilde{f}$ are, in general, not isometric.*

Proof. Since $f_{\mathbb{L}_3}(= f^\lambda_{\mathbb{L}_3}|_{\lambda=1})$ and $\tilde{f}_{\mathbb{L}_3}(= \tilde{f}^\lambda_{\mathbb{L}_3}|_{\lambda=1})$ are isometric by a rigid motion in $\mathbb{L}_3$, the isometry between these two surfaces lifts to the level of frames $F = F^\lambda|_{\lambda=1}$ and $\tilde{F} = \tilde{F}^\lambda|_{\lambda=1}$ as $\tilde{F} = MFk$, where M is a z-independent $\mathrm{SU}_{1,1}$-valued matrix and k is a U_1-valued matrix. After introducing the loop parameter we obtain the relation

$$(1.2) \qquad\qquad \tilde{F}^\lambda = M^\lambda F^\lambda k.$$

Note that M^λ is a $(\Lambda\mathrm{SU}_{1,1})_\sigma$-valued matrix and satisfies $M^\lambda|_{\lambda=1} = M$. Then it is easy to see that $\tilde{f}^\lambda_{\mathbb{L}_3}$ and $f^\lambda_{\mathbb{L}_3}$ satisfy the relation:

$$(1.3) \qquad\qquad \tilde{f}^\lambda_{\mathbb{L}_3} = M^\lambda f^\lambda_{\mathbb{L}_3}(M^\lambda)^{-1} - i\lambda(\partial_\lambda M^\lambda)(M^\lambda)^{-1}.$$

Now a straightforward computation shows that the corresponding two minimal surfaces f^λ and $\tilde{f}^\lambda$ have the following relation:

$$(1.4) \quad \tilde{f}^\lambda = \left(\mathrm{Ad}(M^\lambda)(f^\lambda_{\mathbb{L}_3}) - X^\lambda\right)^o - \frac{1}{2}\{\mathrm{Ad}(M^\lambda)(i\lambda\partial_\lambda f^\lambda_{\mathbb{L}_3}) + [X^\lambda, \mathrm{Ad}(M^\lambda)(f^\lambda_{\mathbb{L}_3})] - Y^\lambda\}^d,$$

where we set

$$(1.5) \qquad\qquad X^\lambda = i\lambda(\partial_\lambda M^\lambda)(M^\lambda)^{-1}, \;\; Y^\lambda = i\lambda(\partial_\lambda X^\lambda),$$

and the superscripts "o" and "d" denote the off-diagonal and diagonal part, respectively. For simplicity of notation we do not distinguish here f^λ in Nil_3 and $\hat{f}^\lambda$ in $\mathfrak{su}_{1,1}$.

We note that for each fixed $\lambda \in \mathbb{S}^1$, the first part of the right-hand side in (1.3) describes a Lorentz transformation and the second part of the right-hand side in (1.3) describes a translation, respectively. We now consider each of the three types of generators separately:

(1) First suppose that $\tilde{f}_{\mathbb{L}_3}$ and $f_{\mathbb{L}_3}$ are isometric by a rotation around x_3-axis (the timelike axis). Since the original transformation M was a rotation, it follows

$$M(= M^\lambda|_{\lambda=1}) = \mathrm{diag}(e^{i\theta}, e^{-i\theta}), \quad \partial_\lambda M^\lambda|_{\lambda=1} = \mathbf{0}.$$

Here 2θ is the angle of rotation. A straightforward computation shows that f and $\tilde{f}$ satisfy the equation $\tilde{f} = \mathrm{Ad}(M)(f) + \frac{1}{2}Y^d$, where the translation term Y can be computed as

$$Y = Y^\lambda|_{\lambda=1} = -\lambda^2(\partial^2_\lambda M^\lambda)(M^\lambda)^{-1}|_{\lambda=1}.$$

Therefore, this one-parameter family consists of isometric minimal surfaces in Nil_3.

(2) Next suppose that $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ are isometric by some translation. Since the original transformation M was a translation, it follows

$$M(= M^\lambda|_{\lambda=1}) = \mathrm{id}, \quad (\partial_\lambda M^\lambda)|_{\lambda=1} \neq \mathbf{0}.$$

Substituting $\lambda = 1$ into (1.4) we see immediately that f and $\tilde{f}$ satisfy the relation $\tilde{f} = f + A$, where $A = A(x_1, x_2)$ is given by

$$(1.6) \qquad\qquad A = -X^o - \frac{1}{2}\left([X, f_{\mathbb{L}_3}] - Y\right)^d,$$

where $X = X^\lambda|_{\lambda=1}$ and $Y = Y^\lambda|_{\lambda=1}$ for X^λ and Y^λ in (1.5). It is clear that Y is independent of x_1 and x_2, the coordinates for f, but x_1 and x_2 enter the commutator. An explicit computation, using the bases stated in Appendix B.3 and the transformation formula stated in Appendix A, now shows that $\tilde{f}$ can be obtained from f by a translation in Nil_3 (with a constant vector, whose coefficients basically are the components of X^o and of Y^d).

(3) Let us finally consider the transformations M given by boosts in $\mathbb{L}_3$. These transformations form a two-parameter family. Since the original transformation was a boost, it follows

$$(1.7) \qquad\qquad M = M^\lambda|_{\lambda=1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \alpha \end{pmatrix}, \quad \partial_\lambda M^\lambda|_{\lambda=1} = \mathbf{0}.$$

Here $\alpha \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $\alpha^2 - |\beta|^2 = 1$ and we obtain

$$\tilde{f} = (\mathrm{Ad}(M)(f_{\mathbb{L}_3}))^o - \frac{1}{2}\left(\mathrm{Ad}(M)(i\lambda\partial_\lambda f_{\mathbb{L}_3}^\lambda|_{\lambda=1}) - Y\right)^d,$$

where $Y = Y^\lambda|_{\lambda=1}$ for Y^λ in (1.5). Now it follows by a straightforward computation that

$$(1.8) \qquad \tilde{f} = \begin{pmatrix} (\alpha^2 + |\beta|^2)r + \alpha\beta\bar{s} - \alpha\bar{\beta}s & -2\alpha\beta p + \alpha^2 q - \beta^2\bar{q} \\ 2\alpha\bar{\beta}p + \alpha^2\bar{q} - \bar{\beta}^2 q & -(\alpha^2 + |\beta|^2)r - \alpha\beta\bar{s} + \alpha\bar{\beta}s \end{pmatrix} + \frac{1}{2}Y^d,$$

where $p, r \in i\mathbb{R}$ and $q, s \in \mathbb{C}$ are functions defined by

$$(1.9) \qquad f_{\mathbb{L}_3}^\lambda|_{\lambda=1} = \begin{pmatrix} p & q \\ \bar{q} & -p \end{pmatrix} \text{ and } -\frac{1}{2}i\lambda(\partial_\lambda f_{\mathbb{L}_3}^\lambda)|_{\lambda=1} = \begin{pmatrix} r & s \\ \bar{s} & -r \end{pmatrix}.$$

Note that the components of the minimal surface f in the basis of Appendix B.3 are given by

$$(x_1, x_2, x_3) = (2\,\mathrm{Im}\,q, -2\,\mathrm{Re}\,q, -2\,\mathrm{Im}\,r).$$

Thus from (1.8) and the action of the isometry group of Nil_3 as described in (A.1), it is easy to see that f and $\tilde{f}$ are in general not isometric, see Remark 1.5 in detail. $\qquad\square$

Remark 1.5. In case (3) of Lemma 1.4, from (1.8) and the action of the isometry group of Nil_3 in (A.1), we see that the f and $\tilde{f}$ are isometric in Nil_3 if and only if there exist some $\theta \in \mathbb{R}$, $(a_1, a_2, a_3) \in \mathbb{R}^3$ such that the following two equations hold:

$$(\alpha^2 + |\beta|^2)x_3 - 4\alpha\,\mathrm{Im}(\beta\bar{s}) - Y^{11} = ax_1 + bx_2 + x_3 + a_3,$$
$$-4\alpha\beta p + i(\beta^2 + \alpha^2)x_1 + (\beta^2 - \alpha^2)x_2 = e^{i\theta}(ix_1 - x_2) + ia_1 - a_2,$$

where $Y^d = \mathrm{diag}(iY^{11}, -iY^{11})$, $a = \frac{1}{2}(a_1 \sin\theta - a_2 \cos\theta)$, $b = \frac{1}{2}(a_1 \sin\theta + a_2 \cos\theta)$, and p, s are purely imaginary, complex valued functions, respectively, defined in (1.9). From these two equations, it is easy to see that they are satisfied for very special minimal surfaces f only.

Remark 1.6. After fixing base points, the Sym-formula establishes a 1-1-relation between spacelike CMC with mean curvature $H = 1/2$ surfaces in $\mathbb{L}_3$ and minimal surfaces in Nil_3. Clearly, the Poincaré group $\mathrm{SU}_{1,1} \ltimes \mathbb{L}_3$ acts on the family of spacelike CMC surfaces in $\mathbb{L}_3$. If we fix base points, we eliminate the action of the translation part of the Poincaré group, reducing the action to the Lorentz group $\mathrm{SU}_{1,1}$.

Via the Sym-formula, the Poincaré group also acts on the family of minimal surfaces in Nil_3. Since we fix base points, we can also eliminate the translation part of the isometry group of Nil_3. So generically, the dimension of the family of minimal surfaces should be three. But from Lemma 1.4, identifying minimal surfaces which are isometric by rotations, we see that two is the highest dimension of any orbit. These orbits are realized by the action of boosts. From (1.7), the set of boosts $\mathcal{B}$ can be computed as

$$\mathcal{B} = \left\{ X \, {}^t\bar{X} \mid X \in \mathrm{SU}_{1,1} \right\}.$$

From this it is clear that $\mathcal{B}$ is the symmetric space $\mathrm{SU}_{1,1}/\mathrm{U}_1$.

1.3. **Bernstein problem.** We will finally present a short alternative proof of the Bernstein problem in Nil_3 using the loop group method. The heart of the proof is the following simple relation between spacelike CMC graphs in $\mathbb{L}_3$ and minimal graphs in Nil_3.

Theorem 1.7. *Every entire, complete, spacelike* CMC *graph in* $\mathbb{L}_3$ *with mean curvature* $H = 1/2$ *and the Hopf differential* $Q_{\mathbb{L}_3} dz^2$ *induces, via the Sym-formula (applied to its associated family), an entire, complete, minimal graph in* Nil_3 *with Abresch-Rosenberg differential* $-Q_{\mathbb{L}_3} dz^2$.

Conversely every entire, complete, minimal graph in Nil_3 *is obtained in this way.*

Proof. Let $g_{\mathbb{L}_3}$ be an entire complete spacelike CMC graph with mean curvature $H = 1/2$ over the (x_1, x_2)-plane in $\mathbb{L}_3$ whose Hopf differential is $Q_{\mathbb{L}_3} dz^2$. Let F^λ be the extended frame of $g_{\mathbb{L}_3}$, see Remark B.3 for the definition, and apply the Sym-formulas of Theorem B.2 to obtain $f^\lambda_{\mathbb{L}_3}$ and f^λ from the same extended frame F^λ. Note that $f^\lambda_{\mathbb{L}_3}$ and f^λ define an associated family of spacelike CMC surfaces in $\mathbb{L}_3$ and minimal surfaces in Nil_3, respectively. Moreover, $f_{\mathbb{L}_3} = f^\lambda_{\mathbb{L}_3}|_{\lambda=1}$ and $g_{\mathbb{L}_3}$ are isometric by some rigid motion in $\mathbb{L}_3$, by the fundamental theorem of surface theory (the mean curvature H, the Hopf differential $Q_{\mathbb{L}_3} dz^2$ and the metric $e^{u_{\mathbb{L}_3}} dz d\bar{z}$ are the same), thus $f_{\mathbb{L}_3}$ is also a entire, complete, spacelike CMC graph, [11, Proposition 1]. From formulas (B.6) and (B.7) we infer by inspection that f^λ and $f^\lambda_{\mathbb{L}_3}$ share the same x_1-, x_2-components. Thus $f = f^\lambda|_{\lambda=1}$ is an entire minimal graph as well. Moreover, from Theorem 1.3 we obtain that $f(= f^\lambda|_{\lambda=1})$ is complete and by Remark B.3 we know that the Abresch-Rosenberg differential is $-Q_{\mathbb{L}_3} dz^2$.

To verify the second statement, let f be an entire, complete, minimal graph in Nil_3 whose Abresch-Rosenberg differential is $Q \, dz^2$ and let F^λ be the extended frame of f and f^λ its associated family from the extended frame F^λ. Then we have $f = f^\lambda|_{\lambda=1}$ up to translation in Nil_3. Moreover, let $f^\lambda_{\mathbb{L}_3}$ be the spacelike CMC surface in $\mathbb{L}_3$ defined by the same extended frame F^λ in (B.6). Note that the Hopf differential of $f^\lambda_{\mathbb{L}_3}$ is $Q^\lambda_{\mathbb{L}_3} dz^2 = -\lambda^{-2} Q \, dz^2$. Since f and $f^\lambda_{\mathbb{L}_3}|_{\lambda=1}$ have the same x_1-, x_2-components, the latter surface is an entire CMC graph in $\mathbb{L}_3$, and thus by [3, p. 415] it is complete. $\qquad\square$

Using Theorem 1.7, it is easy to give the proof of the solution to the Bernstein problem.

Theorem 1.8. *Let $Q\,dz^2$ be a holomorphic quadratic differential on $\mathbb{D}$ or $M = \mathbb{C}$ with $Q \not\equiv 0$. Then the following statements hold:*

   (1) *There exists a two-parameter family of entire, complete, minimal graphs in Nil_3, whose Abresch-Rosenberg differential is $Q\,dz^2$.*
   (2) *Any two members of this two-parameter family are generically non-congruent.*
   (3) *Each member of this two-parameter family is induced via the Sym-formula by (the associated family of) an entire, complete, spacelike CMC graph in $\mathbb{L}_3$ with the Hopf differential $-Q\,dz^2$.*

Proof. First we note that it is known that for a given holomorphic quadratic differential $Q_{\mathbb{L}_3}dz^2$ on $\mathbb{D}$ or $\mathbb{C}$, there exists a unique entire complete spacelike CMC graph $g_{\mathbb{L}_3}$ over the (x_1, x_2)-plane in $\mathbb{L}_3$ whose Hopf differential is $Q_{\mathbb{L}_3}dz^2$, [11, 12]. Here "unique" means that any other such spacelike CMC graph whose Hopf differential is $Q_{\mathbb{L}_3}dz^2$ is isometric to $g_{\mathbb{L}_3}$ by an isometry of $\mathbb{L}_3$. We normalize the mean curvature of $g_{\mathbb{L}_3}$ as $H = 1/2$ and set $Q_{\mathbb{L}_3}dz^2 = -Q\,dz^2$, where $Q\,dz^2$ is the quadratic differential satisfying the condition in the Theorem. Let F^λ be the extended frame of $g_{\mathbb{L}_3}$, see Remark B.3 for the definition, and apply the Sym formulas of Theorem B.2 to obtain $f_{\mathbb{L}_3}^\lambda$ and f^λ from the same extended frame F^λ. From Theorem 1.7 we know that $f_{\mathbb{L}_3} = f_{\mathbb{L}_3}^\lambda|_{\lambda=1}$ and $f = f^\lambda|_{\lambda=1}$ are complete entire graphs. From the construction it is clear that f is a minimal surface. Moreover the Abresch-Rosenberg differential of f is $Q\,dz^2$.

We now consider a spacelike CMC surface $\tilde{g}_{\mathbb{L}_3}$ isometric to $g_{\mathbb{L}_3}$ in $\mathbb{L}_3$. Then as explained in the proof of Lemma 1.4, the extended frame $\tilde{F}^\lambda$ of $\tilde{g}_{\mathbb{L}_3}$ satisfies $\tilde{F}^\lambda = M^\lambda F^\lambda k$ for some z-independent $(\Lambda\mathrm{SU}_{1,1})_\sigma$-valued matrix M^λ and a U_1-valued matrix k, in particular independent of λ. For the associated family $\tilde{f}_{\mathbb{L}_3}^\lambda$ of $\tilde{g}_{\mathbb{L}_3}$ which is defined by the Sym formula (B.6) from the extended frame $\tilde{F}^\lambda$, we see that $\tilde{f}_{\mathbb{L}_3} = \tilde{f}_{\mathbb{L}_3}^\lambda|_{\lambda=1}$ and $\tilde{g}_{\mathbb{L}_3}$ are isometric. Thus $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ are isometric, and again from [11, Proposition 1] we obtain that $\tilde{f}_{\mathbb{L}_3}$ is an entire, complete, spacelike CMC graph. Let $\tilde{f}^\lambda$ be the corresponding associated family of minimal surfaces in Nil_3 which is defined by the Sym formula (B.7) from the extended frame $\tilde{F}^\lambda$. Then using the argument in Theorem 1.7, we see that $\tilde{f} = \tilde{f}^\lambda|_{\lambda=1}$ is an entire, complete minimal graph in Nil_3. Note that the Abresch-Rosenberg differential of $\tilde{f}$ is also $Q\,dz^2$.

We now apply Lemma 1.4. If the isometry $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ is of case (1) or (2), then $\tilde{f}(= \tilde{f}^\lambda|_{\lambda=1})$ is congruent to $f(= f^\lambda|_{\lambda=1})$. However, if the isometry of $f_{\mathbb{L}_3}$ and $\tilde{f}_{\mathbb{L}_3}$ is of case (3), then $\tilde{f}$ is in general non-congruent to f. In particular, the case (3) in Lemma 1.4 corresponds to a two-parameter family of boosts in $\mathbb{L}_3$. Therefore, for an entire, complete, spacelike CMC surface, there exists a two-parameter family of non-congruent complete minimal graphs in Nil_3 which have the same Abresch-Rosenberg differential Qdz^2. $\square$

Remark 1.9. In [7], the two-parameter family of an entire, complete, minimal graph was obtained by the choice of the initial condition for a nonlinear partial differential equation. The solution corresponds to ϕ_3, that is, the e_3-component of $f^{-1}f_z$ and the initial condition is the initial value $\phi_3(z_*)$ for some base point z_* in $\mathbb{C}$ or $\mathbb{D}$. In our setting, this freedom naturally appears as the two-parameter family of boosts in $\mathbb{L}_3$: As we see from the proof of

Theorem 1.8, two minimal surfaces f^λ and $\tilde{f}^\lambda$ satisfy the relation (1.4). Set

$$M^\lambda|_{\lambda=1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \alpha \end{pmatrix} \in \mathrm{SU}_{1,1}, \quad (f^\lambda)^{-1}f_z^\lambda|_{\lambda=1} = \sum_{k=1}^{3}\phi_k e_k, \text{ and } (\tilde{f}^\lambda)^{-1}\tilde{f}_z^\lambda|_{\lambda=1} = \sum_{k=1}^{3}\tilde{\phi}_k e_k,$$

where α is real. Then a straightforward computation (using the proof of [5, Theorem 6.1]) shows

$$(1.10) \qquad \tilde{\phi}_3 = (\alpha^2 + |\beta|^2)\phi_3 + 2i\,\mathrm{Re}(\alpha\beta)\phi_1 + 2i\,\mathrm{Im}(\alpha\beta)\phi_2.$$

From this expression, it is clear that our two-parameter family of boosts induces a freedom of the initial condition of ϕ_3 which is naturally parametrized by $\mathbb{C}$.

Corollary 1.10. *The associated family of every entire, complete, minimal graph in Nil_3 with a given Abresch-Rosenberg differential $Q\,dz^2$ is a family of entire, complete, minimal graphs in Nil_3 with the Abresch-Rosenberg differential $\lambda^{-2}Qdz^2$ ($\lambda \in \mathbb{S}^1$). Moreover, within a given associated family, complete minimal graphs have the same support $h\,(dz)^{1/2}(d\bar{z})^{1/2}$.*

Proof. From the proof of Theorem 1.8, it is clear that for an entire minimal graph $f^\lambda|_{\lambda=1}$ all members of its associated family of minimal surfaces have the Abresch-Rosenberg differential $\lambda^{-2}Qdz^2$ ($\lambda \in \mathbb{S}^1$) and the same support $h\,(dz)^{1/2}(d\bar{z})^{1/2}$. To prove that the minimal surfaces in the associated family are graphs, we consider the spacelike CMC surfaces $f_{\mathbb{L}_3}^\lambda$ given in the proof of Theorem 1.8. Then, since $f_{\mathbb{L}_3}^\lambda|_{\lambda=1}$ is entire, it is complete [3, p. 415] and thus the spacelike CMC surfaces $f_{\mathbb{L}_3}^\lambda$ in the associated family are also complete. They are in fact isometric to $f_{\mathbb{L}_3}^\lambda|_{\lambda=1}$. Note that the complete metric is given by $h^2\,dzd\bar{z}$, see (1.1). Therefore, by [11, Proposition 1], all $f_{\mathbb{L}_3}^\lambda$ are entire graphs, and thus the corresponding minimal surfaces f^λ in Nil_3 are also entire graphs. The completeness of the associated family follows from Lemma 1.1. $\qquad\square$

Remark 1.11 (Canonical examples). In [7], all entire, complete, minimal vertical graphs are called the *canonical examples*.

Appendix A. Isometry group of the three-dimensional Heisenberg group

A.1. The identity component $\mathrm{Iso}_\circ(\mathrm{Nil}_3)$ of the isometry group of Nil_3 is the semi-direct product $\mathrm{Nil}_3 \ltimes \mathrm{SO}_2$. If we identify Nil_3 with $\mathbb{C} \times \mathbb{R}$ and SO_2 with U_1, respectively, then the action of $\mathrm{Nil}_3 \ltimes \mathrm{SO}_2\,(\cong (\mathbb{C} \times \mathbb{R}) \ltimes \mathrm{U}_1)$ is given by

$$(A.1) \quad ((\alpha = a_1 + ia_2,\ a_3),\ e^{i\theta}) \cdot (z = x_1 + ix_2,\ x_3) = \left(e^{i\theta}z + \alpha,\ x_3 + \tfrac{1}{2}\,\mathrm{Im}(\bar{\alpha}e^{i\theta}z) + a_3\right),$$

where $\theta, a_3, x_3 \in \mathbb{R}$ and $\alpha, z \in \mathbb{C}$. Here (x_1, x_2, x_3) is a coordinate system of Nil_3, θ is a rotation angle and (a_1, a_2, a_3) is a translation vector. The Heisenberg group Nil_3 is represented by $(\mathrm{Nil}_3 \ltimes \mathrm{SO}_2)/\mathrm{SO}_2$ as a naturally reductive homogeneous space. One can see that this homogeneous space is not Riemannian symmetric.

Appendix B. Basic Results

B.1. **Basic notation.** Let Nil_3 be the three-dimensional Heisenberg group with the bundle curvature $\tau = 1/2$ and let $f : M \to \mathrm{Nil}_3$ a conformal immersion of a Riemann surface M into Nil_3. Denote the orthonormal basis of the Lie algebra of Nil_3 by $\{e_1, e_2, e_3\}$. Then

the Maurer-Cartan form $f^{-1}df$ can be expanded as $f^{-1}df = (f^{-1}f_z)dz + (f^{-1}f_{\bar{z}})d\bar{z}$ with $f^{-1}f_z = \sum_{k=1}^{3}\phi_k e_k$ and $f^{-1}f_{\bar{z}} = \overline{f^{-1}f_z} = \sum_{k=1}^{3}\bar{\phi}_k e_k$. Here $(z = x + iy)$ are conformal coordinates, $\bar{z} = x - iy$ is its complex conjugate, and the subscripts z and $\bar{z}$ denote the partial differentiations with respect to z and $\bar{z}$, respectively. Moreover ϕ_k is a complex-valued function and $\bar{\phi}_k$ is the its complex conjugate function. Since f is a conformal immersion, it is easy to see that $\phi_k (k = 1, 2, 3)$ satisfy $\sum_{k=1}^{3}\phi_k^2 = 0$ and $\sum_{k=1}^{3}|\phi_k|^2 = \frac{1}{2}e^u \neq 0$. We note that the induced metric of f is given by $ds^2 = e^u dz d\bar{z}$. Then using the *generating spinors* ψ_1 and ψ_2, the first equation can be solved by

$$\phi_1 = (\overline{\psi_2})^2 - \psi_1^2, \quad \phi_2 = i((\overline{\psi_2})^2 + \psi_1^2), \quad \phi_3 = 2\psi_1\overline{\psi_2}.$$

Then the condition $\sum_{k=1}^{3}|\phi_k|^2 = e^u/2$ is equivalent with $e^u = 4(|\psi_1|^2 + |\psi_2|^2)^2$. Let N be the positively oriented unit normal vector field along f and denote an unnormalized normal vector field L by $L = e^{u/2}N$. We define the *support* $h(dz)^{1/2}(d\bar{z})^{1/2}$ by $h = \langle f^{-1}L, e_3 \rangle$. Then it is easy to compute h by the generating spinors ψ_1 and ψ_2: $h = \langle f^{-1}L, e_3 \rangle = 2(|\psi_1|^2 - |\psi_2|^2)$. Moreover, let $e^{w/2}$ and $Q\,dz^2 = 4B\,dz^2$ be the Dirac potential and the Abresch-Rosenberg differential [1], given by

$$e^{w/2} = \mathcal{U} = \mathcal{V} = -\frac{H}{2}e^{u/2} + \frac{i}{4}h, \quad \text{and } B = \frac{2H+i}{4}\left(\langle f_{zz}, N \rangle + \frac{\phi_3^2}{2H+i}\right),$$

respectively. It is known that the vector of generating spinors $\tilde{\psi} = (\psi_1, \psi_2)$ satisfies the so-called "linear spinor system" [5]:

(B.1) $$\tilde{\psi}_z = \tilde{\psi}\tilde{U}, \quad \tilde{\psi}_{\bar{z}} = \tilde{\psi}\tilde{V},$$

where

$$\tilde{U} = \begin{pmatrix} \frac{1}{2}w_z + \frac{1}{2}H_z e^{-w/2+u/2} & -e^{w/2} \\ Be^{-w/2} & 0 \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} 0 & -\bar{B}e^{-w/2} \\ e^{w/2} & \frac{1}{2}w_{\bar{z}} + \frac{1}{2}H_{\bar{z}}e^{-w/2+u/2} \end{pmatrix}.$$

We note that the second column of the first equation and the first column of the second equation together are the nonlinear Dirac equations, that is,

$$\partial_z\psi_2 = -\mathcal{U}\psi_1, \quad \partial_{\bar{z}}\psi_1 = \mathcal{V}\psi_2,$$

where $\mathcal{U} = \mathcal{V} = e^{w/2}$.

B.2. **Flat connections.** From now on we assume that the unit normal $f^{-1}N$ is upward, that is, the e_3-component of $f^{-1}N$ is positive. Since $f^{-1}N$ is upward, there is a stereographic projection π of the unit normal $f^{-1}N$ from the south pole to the unit disk in $\mathbb{R}^2$. We denote the map $\pi \circ f^{-1}N$ by g and call g the *normal Gauss map*. Then it is easy to see that g can be represented by the generating spinors as

$$g = \frac{\psi_2}{\overline{\psi_1}}.$$

We now define the family of Maurer-Cartan forms α^λ as $\alpha^\lambda = \tilde{U}^\lambda dz + \tilde{V}^\lambda d\bar{z}$ with

(B.2) $$\tilde{U}^\lambda = \begin{pmatrix} \frac{1}{4}w_z + \frac{1}{2}H_z e^{-w/2+u/2} & -\lambda^{-1}e^{w/2} \\ \lambda^{-1}Be^{-w/2} & -\frac{1}{4}w_z \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} -\frac{1}{4}w_{\bar{z}} & -\lambda\bar{B}e^{-w/2} \\ \lambda e^{w/2} & \frac{1}{4}w_{\bar{z}} + \frac{1}{2}H_{\bar{z}}e^{-w/2+u/2} \end{pmatrix}.$$

Then minimal surfaces in Nil_3 are characterized in terms of the normal Gauss map as follows.

Theorem B.1 (Theorem 5.3 in [5])**.** *Let $f : \mathbb{D} \to \mathrm{Nil}_3$ be a conformal immersion which is nowhere vertical and α^λ the 1-form defined in (B.2). Moreover, assume that the unit normal $f^{-1}N$ is upward. Then the following statements are equivalent:*

(1) *f is a minimal surface.*
(2) *$d + \alpha^\lambda$ is a family of flat connections on $\mathbb{D} \times \mathrm{SU}_{1,1}$.*
(3) *The normal Gauss map g for f is a non-conformal harmonic map into the hyperbolic two-space $\mathbb{H}^2$.*

Definition 1. Let f be a minimal surface in Nil_3 and F^λ a $(\Lambda\mathrm{SU}_{1,1})_\sigma$-valued solution to the equation $(F^\lambda)^{-1}dF^\lambda = \alpha^\lambda$ such that

(B.3)
$$F^\lambda|_{\lambda=1} = \frac{1}{\sqrt{|\psi_1|^2 - |\psi_2|^2}} \begin{pmatrix} \sqrt{i}^{-1}\psi_1 & \sqrt{i}^{-1}\psi_2 \\ \sqrt{i}\,\overline{\psi_2} & \sqrt{i}\,\overline{\psi_1} \end{pmatrix}.$$

Then F is called an *extended frame* of the minimal surface f.

B.3. Sym-formula. First we note that the (multiple of the) Killing form $\langle A, B \rangle = 4\,\mathrm{Tr}\,AB$ induces a Lorentz metric on $\mathfrak{su}_{1,1}$. Thus we regard $\mathfrak{su}_{1,1}$ as the Minkowski 3-space. The basis

(B.4)
$$\mathcal{E}_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \mathcal{E}_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{E}_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

is an orthonormal basis of $\mathfrak{su}_{1,1}$ with timelike vector $\mathcal{E}_3$.

The timelike vector $\mathcal{E}_3$ generates the rotation group SO_2 which acts isometrically on $\mathbb{L}_3$ by rotations around the x_3-axis. On the other hand, the isometries $\exp(t\mathcal{E}_1)$ and $\exp(t\mathcal{E}_2)$ are called *boosts*.

Now we identify the Lie algebra $\mathfrak{nil}_3$ of Nil_3 with the Lie algebra $\mathfrak{su}_{1,1}$ as a *real vector space*. A linear isomorphism $\Xi : \mathfrak{su}_{1,1} \to \mathfrak{nil}_3$ is then given by

(B.5)
$$\mathfrak{su}_{1,1} \ni x_1\mathcal{E}_1 + x_2\mathcal{E}_2 + x_3\mathcal{E}_3 \longmapsto x_1 e_1 + x_2 e_2 + x_3 e_3 \in \mathfrak{nil}_3.$$

Note that the linear isomorphism Ξ is not a Lie algebra isomorphism. Next we consider the exponential map $\exp : \mathfrak{nil}_3 \to \mathrm{Nil}_3$

$$\exp(x_1 e_1 + x_2 e_2 + x_3 e_3) = e^{x_1} E_{11} + \sum_{i=2}^{4} E_{ii} + x_1 E_{23} + (x_3 + \frac{1}{2}x_1 x_2)E_{24} + x_2 E_{34},$$

where E_{ij} is a 4 by 4 matrix with the ij-entry equal to 1, and all other entries equal to 0. Here we imbed Nil_3 into $\mathrm{GL}_4\mathbb{R}$ by $\iota : \mathrm{Nil}_3 \to \mathrm{GL}_4\mathbb{R}$, $\iota(x_1 + x_2 + x_3) = e^{x_1} E_{11} + \sum_{i=2}^{4} E_{ii} + x_1 E_{23} + (x_3 + \frac{1}{2}x_1 x_2)E_{24} + x_2 E_{34}$. We define a smooth bijection $\Xi_{\mathrm{nil}} : \mathfrak{su}_{1,1} \to \mathrm{Nil}_3$ by $\Xi_{\mathrm{nil}} := \exp \circ \Xi$. Under this identification $\mathrm{Nil}_3 = \mathfrak{su}_{1,1}$, $\mathrm{SO}_2 = \{\exp(t\,\mathcal{E}_3)\}_{t\in\mathbb{R}}$ acts isometrically on Nil_3 as rotations around x_3-axis.

In what follows we will take derivatives for functions of λ. Note that for $\lambda = e^{i\theta} \in \mathbb{S}^1$, we have $\partial_\theta = i\lambda\partial_\lambda$.

Theorem B.2 (Theorem 6.1 in [5])**.** *For the extended frame F^λ of some minimal surface f, define maps $f^\lambda_{\mathbb{L}_3}$ and $N^\lambda_{\mathbb{L}_3}$ respectively by*

(B.6)
$$f^\lambda_{\mathbb{L}_3} = -i\lambda(\partial_\lambda F^\lambda)(F^\lambda)^{-1} - N^\lambda_{\mathbb{L}_3} \quad \text{and} \quad N^\lambda_{\mathbb{L}_3} = \frac{i}{2}\,\mathrm{Ad}(F^\lambda)\sigma_3.$$

Moreover, define a map $f^\lambda : \mathbb{D} \to \mathrm{Nil}_3$ *by* $f^\lambda := \Xi_{\mathrm{nil}} \circ \hat{f}^\lambda$ *with*

(B.7)
$$\hat{f}^\lambda = (f^\lambda_{\mathbb{L}_3})^o - \frac{i}{2}\lambda(\partial_\lambda f^\lambda_{\mathbb{L}_3})^d,$$

where the superscripts "o" and "d" denote the off-diagonal and diagonal part, respectively. Then, for each $\lambda \in \mathbb{S}^1$, *the following statements hold*:

(1) *The map* $f^\lambda_{\mathbb{L}_3}$ *is a spacelike* CMC *surface with mean curvature* $H = 1/2$ *in* $\mathbb{L}_3$ *and* $N^\lambda_{\mathbb{L}_3}$ *is the timelike unit normal vector of* $f^\lambda_{\mathbb{L}_3}$.
(2) *The map* f^λ *is a minimal surface in* Nil_3 *and* $N^\lambda_{\mathbb{L}_3}$ *is the normal Gauss map of* f^λ. *In particular,* $f^\lambda|_{\lambda=1}$ *gives the original minimal surface* f *up to translation.*

Remark B.3.

(1) It is known that the Maurer-Cartan form $\alpha^\lambda = \tilde{U}^\lambda dz + \tilde{V}^\lambda d\bar{z}$ in (B.2) with $H = 0$ and $\lambda = 1$ is the Maurer-Cartan form of a spacelike CMC surface with mean curvature $H = 1/2$, the Hopf differential $Q_{\mathbb{L}_3} dz^2 = -4B\,dz^2$ and the metric $h^2 dz d\bar{z}$, see [2, Lemma 3.1].[1] Any $(\Lambda \mathrm{SU}_{1,1})_\sigma$-valued solution F^λ of $(F^\lambda)^{-1} dF^\lambda = \alpha^\lambda$ is called the *extended frame* of a spacelike CMC surface in $\mathbb{L}_3$.
(2) The Hopf differential of the spacelike CMC surface $f^\lambda_{\mathbb{L}_3}$ in Theorem B.2 can be computed as $Q^\lambda_{\mathbb{L}_3} dz^2 = -4\lambda^{-2} B\,dz^2$, where $Q^\lambda dz^2 = 4\lambda^{-2} B dz^2$ is the Abresch-Rosenberg differential of the minimal surface f^λ in Nil_3.
(3) Note that in Theorem B.2 the choice of coordinates is free. We will therefore apply this result to graph coordinates as well as to conformal coordinates without further mentioning.

In the following Corollary, we compute the Abresch-Rosenberg differential Bdz^2 for the 1-parameter family f^λ in Theorem B.2 and it implies that the family f^λ actually defines the associated family.

Corollary B.4. *Let* f^λ *be the family of minimal surfaces in* Nil_3 *defined by* (B.7). *Then* f^λ *preserves the mean curvature* $(= 0)$ *and the support. Moreover, the Abresch-Rosenberg differential* $Q^\lambda dz^2$ *for* f^λ *is given by* $Q^\lambda dz^2 = 4\lambda^{-2} B dz^2$, *where* $Q\,dz^2 = 4B\,dz^2$ *is the Abresch-Rosenberg differential for* $f^\lambda|_{\lambda=1}$. *Therefore* $\{f^\lambda\}_{\lambda \in \mathbb{S}^1}$ *is the associated family of the minimal surface* $f^\lambda|_{\lambda=1}$.

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